Arithmetic Saturation and Pathological Satisfaction

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Recursive Saturation / Resplendency

Fix a finite language $\mathcal{L}$. Recall the following definitions (Barwise-Schlipf, 1976):

1. A first order structure $\mathfrak{A}$ is recursively saturated if, whenever $p(x)$ is a computable, consistent type (possibly including finitely many parameters from $\mathfrak{A}$), there is $c \in \mathfrak{A}$ realizing $p(x)$.
2. A first order structure $\mathfrak{A}$ is resplendent if, whenever $R \notin \mathcal{L}$ is a new relation symbol, $\bar{a} \in \mathfrak{A}$ is a tuple and $\phi(\bar{y}) \in \mathcal{L} \cup \{R\}$ is a formula, if $\mathrm{Th}(\mathfrak{A}, \bar{a}) \cup \{\phi(\bar{a})\}$ is consistent, then $\mathfrak{A}$ has an expansion $(\mathfrak{A}, R) \models \phi(\bar{a})$.

Theorem (Barwise-Schlipf)

A countable structure is recursively saturated if and only if it is resplendent.

Smorynski (1981) improved this to "chronic resplendency."

Satisfaction classes

Theorem

Let $\mathcal{M} \models \mathrm{PA}$ be countable. Then $\mathcal{M}$ has a full satisfaction class $S \subseteq M^2$ if and only if $\mathcal{M}$ is recursively saturated.

- Satisfaction class: for each formula ϕ, assignment α, if $\mathcal{M} \models \phi[\alpha]$, then $(\phi, \alpha) \in S$. (Identify formulas with codes)
- Satisfies Tarski's compositional axioms for satisfaction.
- Full: for each $\phi \in \mathrm{Form}^{\mathcal{M}}$, α, either $(\phi, \alpha) \in S$ or $(\neg\phi, \alpha) \in S$.
- $\Longleftarrow$: Kotlarski, Krajewski, Lachlan (1981)
- $\Longrightarrow$: Lachlan (1981).
- Enayat, Visser (2015): perspicuous model-theoretic proof (of KKL).

(After this: assume all models of PA in this talk are countable and recursively saturated.)

Induction

Definition

Let $\mathcal{M} \models \mathsf{PA}$. $X \subseteq M$ is inductive if the expansion $(\mathcal{M}, X) \models \mathsf{PA}^*$: that is, if the expansion satisfies induction in the language $\mathcal{L}_{\mathsf{PA}} \cup \{X\}$.

- Blur lines: truth predicates / satisfaction classes
- CT^- (theory of a full, compositional truth predicate) is conservative over PA: if $\phi \in \mathcal{L}_{\mathsf{PA}}$, $\mathsf{CT}^- \vdash \phi$ if and only if $\mathsf{PA} \vdash \phi$.
- CT is the theory $\mathsf{CT}^- +$ "T is inductive"
- CT is not conservative over PA: $\mathsf{CT} \vdash \mathrm{Con}(\mathsf{PA})$
- CT_0: $\mathsf{CT}^- +$ "T is Δ_0-inductive" also proves $\mathrm{Con}(\mathsf{PA})$.

Disjunctive Correctness

Definition

Let $c \in M$, $\langle \phi_i : i \leq c \rangle$ be a (coded) sequence of sentences in $\mathcal{M}$. Then we define $\bigvee_{i \leq c} \phi_i$ inductively:

- $\bigvee_{i \leq 0} \phi_i = \phi_0$, and
- $\bigvee_{i \leq n+1} \phi_i = \bigvee_{i \leq n} \phi_i \vee \phi_{n+1}$.

DC is the principle of disjunctive correctness:

$$\forall c \forall \langle \phi_i : i \leq c \rangle T(\bigvee_{i \leq c} \phi_i) \leftrightarrow \exists i \leq c T(\phi_i).$$

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Theorem (Enayat-Pakhomov)

$\mathrm{CT}^- + \mathrm{DC} = \mathrm{CT}_0$.

DC-out vs DC-in

- DC-out: $T(\bigvee_{i \leq c} \phi_i) \rightarrow \exists i \leq c T(\phi_i)$.
- DC-in: $\exists i \leq c T(\phi_i) \rightarrow T(\bigvee_{i \leq c} \phi_i)$.

Theorem (Cieśliński, Łełyk, Wcisło)

- $\mathrm{CT}^- + \mathrm{DC}$-*out is not conservative over* PA. *(in fact, it is equivalent to* CT_0*).*
- $\mathrm{CT}^- + \mathrm{DC}$-*in is conservative over* PA.

Disjunctive Triviality

Idea (for conservativity of DC-in): every $\mathcal{M} \models \mathrm{PA}$ countable has an elementary extension $\mathcal{N}$ with an expansion to CT^- that is disjunctively trivial.

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Question

Does every countable, recursively saturated model of PA *have a disjunctively trivial expansion?*

Intuitively: seems like it should follow from the existence of disjunctively trivial elementary extensions using resplendence?

Slogan

- Preventing pathologies requires (some) induction.
- Conservative truth theories necessarily carry pathologies.

Idempotent disjunctions

Instead of considering all disjunctions, we will study idempotent disjunctions (disjunctions of a single sentence θ).

Question

Let $\mathcal{M} \models \mathsf{PA}$ (countable, recursively saturated). Fix a false (standard) sentence θ (ex: $0 = 1$). For which sets X must there be a satisfaction class S such that $X = \{c : (\bigvee_{i \leq c} \theta, \emptyset) \in S\}$?

Clearly:

- X is closed under successors, predecessors.
- $0 \notin X$, therefore $\omega \cap X = \emptyset$.

What else?

Separability

Recall that models of PA have the ability to code ($\mathcal{M}$-)finite sets and sequences. For $\mathcal{M} \models \mathrm{PA}$, $a, b \in M$:

- $(a)_b$ denotes the b-th element of the ($\mathcal{M}$-finite) sequence coded by a.
- $\mathcal{M} \models a \in b$ if a is in the ($\mathcal{M}$-finite) set coded by b.

Definition

Let $A \subseteq D \subseteq M$. A is separable from D if for each $a \in M$ such that $\{(a)_n : n \in \omega\} \subseteq D$, there is $c \in M$ such that for each $n \in \omega$, $(a)_n \in A$ if and only if $n \in c$. We say a set $X \subseteq M$ is separable if it is separable from M.

It turns out, this is all we need!

Separability Results

We will see:

- If $X \subseteq M$ is separable, disjoint from ω, and is closed under successors and predecessors, then $\mathcal{M}$ has a full satisfaction class such that X is the set of lengths of true disjunctions of $0 = 1$; and,
- If D is any set of sentences and $A = \{\phi \in D : (\phi, \emptyset) \in S\}$, then A is separable from D.

Both of these are, essentially, due to unpublished work by Jim Schmerl (Sent to A. Enayat in private communication, 2012).

Exercise

Proposition

Suppose $D = \{\bigvee_{i \le c} (0 = 1) : c \in M\}$ *and* $A = \{\bigvee_{i \le c} (0 = 1) : c \in X\}$ *for some set* X*. Then* A *is separable from* D *if and only if* X *is separable (from* M*).*

$\Longrightarrow$: Immediate from the definitions.

$\Longleftarrow$: Let $a \in M$ be such that for each $n \in \omega$, $(a)_n$ is a disjunction of $(0 = 1)$. Let $b \in M$ be such that for each $n \in \omega$, $(a)_n = \bigvee_{i \le (b)_n} (0 = 1)$ (use saturation of $\mathcal{M}$ to find b). Then since X is separable, there is c such that $(b)_n \in X$ if and only if $n \in c$. This c shows A is separable from D.

Arithmetic Saturation

Definition

$\mathcal{M}$ is arithmetically saturated if whenever $a, b \in M$ and $p(x, b)$ is a consistent type that is arithmetic in $\mathrm{tp}(a)$ is realized in $\mathcal{M}$.

Folklore: $\mathcal{M}$ is arithmetically saturated if it is recursively saturated and ω is a strong cut: that is, for each a there is $c > \omega$ such that for each $n \in \omega$, $(a)_n \in \omega$ if and only if $(a)_n < c$. Exercise: ω is a strong cut iff it is separable.

Corollary

Let $\mathcal{M}$ be countable and recursively saturated. Then $\mathcal{M}$ is arithmetically saturated if and only if it has a disjunctively trivial expansion to CT^-.

Nonstandard sentences

What's special about disjunctions? Nothing. Fix θ. We consider the following examples of nonstandard iterates of θ.

- $\bigvee_{i \leq c} \theta := (\bigvee_{i \leq c-1} \theta) \vee \theta$
- $\bigwedge_{i \leq c} \theta := (\bigwedge_{i \leq c-1} \theta) \wedge \theta$
- $\bigvee_{i \leq c}^{\text{bin}} \theta := (\bigvee_{i \leq c-1}^{\text{bin}} \theta) \vee (\bigvee_{i \leq c-1}^{\text{bin}} \theta)$
- $(\forall y)^c \theta := \forall y [(\forall y)^{c-1} \theta]$
- $(\neg\neg)^c \theta := \neg\neg[(\neg\neg)^{c-1} \theta]$

All of the above are formed by taking θ, the $(c-1)$-st iterate of θ, and combining them syntactically in a predetermined way. **Both results generalize to all of these situations!**

Generalization

Fix θ an atomic sentence. Let $\Phi(p, q)$ be a finite propositional "template" (essentially a propositional formula with variables p, q, but we allow quantifiers over dummy variables) such that:

- q appears in $\Phi(p, q)$,
- if $\mathcal{M} \models \theta$, then $\Phi(\top, q)$ is equivalent to q, and
- if $\mathcal{M} \models \neg\theta$, then $\Phi(\bot, q)$ is equivalent to q.

Define $F : M \to \mathrm{Sent}^{\mathcal{M}}$ by $F(0) = \theta$ and $F(x + 1) = \Phi(\theta, F(x))$.
We say such an F is a local idempotent sentential operator for θ.
Φ is called a *template* for F.

Examples

- $\Phi(p,q) = q \vee p$. Then $F(x) = \bigvee_{i \leq x} \theta$.
- $\Phi(p,q) = q \wedge q$. Then $F(x) = \bigwedge_{i \leq x}^{\text{bin}} \theta$. (Binary conjunctions).
- $\Phi(p,q) = \neg\neg q$. Then $F(x) = (\neg\neg)^x \theta$.
- $\Phi(p,q) = (\forall y) q$. Then $F(x) = (\forall y)^x \theta$.

Separability Theorem 1

Theorem

Fix θ and a local idempotent sentential operator F. Let $X \subseteq M$ be separable, closed under successors and predecessors, and for each $n \in \omega$, $n \in X$ if and only if $\mathcal{M} \models \theta$. Then $\mathcal{M}$ has a full satisfaction class S such that $X = \{x : S(F(x), \emptyset)\}$.

Proof sketch

- For $Y \subseteq \text{Form}^{\mathcal{M}}$, $\text{Cl}(Y)$ is the smallest $Z \supseteq Y$ closed under immediate subformulas.
- Y is *finitely generated* if $Y = \text{Cl}(Y')$ for some finite Y'.

Main part of construction: suppose Y is finitely generated and S is a full satisfaction class such that $(\mathcal{M}, S)$ is recursively saturated and whenever $F(x) \in Y$, then $x \in X$ if and only if $S(F(x), \emptyset)$. Let $Y' \supseteq Y$ be finitely generated. Show that the following theory is consistent:

- S' is a full satisfaction class (Enayat-Visser lemma),
- $S \restriction Y = S' \restriction Y$,
- $\{S'(F(x), \alpha) : F(x) \in Y' \text{ and } x \in X\}$.

Using the facts that Y, Y' are finitely generated and X is separable, the above can be expressed recursively. Apply resplendency.

Separability Theorem 2

Theorem

Let D be any set of sentences, S a full satisfaction class for $\mathcal{M}$, and $A = \{\phi \in D : S(\phi, \emptyset)\}$. Then A is separable from D.

Proof sketch: Stuart Smith's Theorem: $\mathcal{M}$ is *definably S-saturated*. That is: if $\langle \phi_i(x) : i \in \omega \rangle$ is coded such that for each $m \in \omega$, there is an assignment α such that for all $i \leq m$, $S(\phi_i, \alpha)$, then there is α such that for all $i \in \omega$, $S(\phi_i, \alpha)$.

Let a be such that $(a)_n \in D$ for all $n \in \omega$, $\phi_i(x)$ the formula $(a)_i \leftrightarrow i \in x$. For each standard m, there is c such that for $i \leq m$, $(a)_i \in A$ if and only if $i \in c$. Apply Smith's result.

Separable cuts

Proposition

Let $I \subseteq_{\text{end}} M$ be a cut. Then the following are equivalent:

1. *I is separable.*
2. *There is no a such that* $I = \sup(\{(a)_n : n \in \omega\} \cap I) = \inf(\{(a)_n : n \in \omega\})$.
3. *For each $a \in M$, there is c such that for each $n \in \omega$, $(a)_n \in I$ if and only if $(a)_n < c$.*

Proof is an exercise.

Existence

If $(\mathcal{M}, I)$ is recursively saturated, then I is separable. (Exercise.)

Proposition

There are separable cuts which are closed under successor but not addition, addition but not multiplication, multiplication but not exponentiation, etc.

Proof (for $+$ but not $\times$): Let $I \subseteq_{\text{end}} M$ be any cut which is closed under addition but not multiplication (ex: $c > \omega$, $I = \sup(\{n \cdot c : n \in \omega\})$). Then by resplendence, there is $J \subseteq_{\text{end}} M$ such that $(\mathcal{M}, J)$ is recursively saturated and J is closed under addition but not multiplication.

Superrational Cuts

R. Kossak (1989) introduced notions of "rational" / "superrational" cuts.

Definition (Kossak 1989)

Let $I \subseteq_{\text{end}} M$.

1. I is *coded by* ω *from below* if there is $a \in M$ such that $I = \sup(\{(a)_i : i \in \omega\})$. I is *coded by* ω *from above* if there is $a \in M$ such that $I = \inf(\{(a)_i : i \in \omega\})$. I is ω*-coded* if it is either coded by ω from below or from above.
2. I is 0-*superrational* if there is $a \in M$ such that one of the following holds:
   - $\text{Def}_0(a) \cap I$ is cofinal in I and for all $b \in M$, $\text{Def}_0(b) \setminus I$ is not coinitial in $M \setminus I$, or,
   - $\text{Def}_0(a) \setminus I$ is coinitial in $M \setminus I$ and for all $b \in M$, $\text{Def}_0(b) \cap I$ is not cofinal in I.

Strength

Theorem

Let $I \subseteq_{\text{end}} M$. The following are equivalent:

1. *I is ω-coded and separable.*
2. *I is 0-superrational.*

Proposition

1. *If ω is a strong cut, then every ω-coded cut is separable.*
2. *If ω is not strong, then every ω-coded cut is not separable.*

Non-local operators

Instead of simply looking at F-iterates of a single θ, what about all F-iterates? (Instead of long idempotent disjunctions of $(0 = 1)$, what about all idempotent disjunctions?)

Fix $\Phi(p, q)$ a finite propositional template such that:

- q appears in $\Phi(p, q)$,
- $p \wedge q \vdash \Phi(p, q)$,
- $\neg p \wedge \neg q \vdash \neg\Phi(p, q)$, and,
- Φ has syntactic depth 1.

Define $F(x, \phi)$ inductively:

- $F(0, \phi) = \phi$.
- $F(x + 1, \phi) = \Phi(\phi, F(x, \phi))$.

We call F an idempotent sentential operator, and say Φ is a template for F.

Accessibility / Additivity

Proposition

Suppose F is an idempotent sentential operator and $\Phi(p, q)$ is a template for F. If p does not appear in Φ, then for any sentence ϕ and any $x, y \in M$, $F(x, F(y, \phi)) = F(x + y, \phi)$.

That is: $(\forall y)^{c_1}[(\forall y)^{c_2}\phi] = (\forall y)^{c_1+c_2}\phi$.

We say F is accessible if p occurs in Φ (then you can "access" ϕ from $F(x, \phi)$); F is additive otherwise.

Additivity

Proposition

Let $I \subseteq_{\text{end}} M$ be a cut. Let F be an additive idempotent sentential operator and S a full satisfaction class such that

$$I = \{x : \forall c < x \forall \phi (S(\phi, \emptyset) \leftrightarrow S(F(c, \phi), \emptyset))\}.$$

Then I is closed under addition.

Proof: Suppose $x \in I$. Let $c < 2x$. Then $\lceil \frac{c}{2} \rceil < x$. For $\phi \in \text{Sent}$, we have

$$S(\phi, \emptyset) \leftrightarrow S(F(\lceil \frac{c}{2} \rceil, \phi), \emptyset) \leftrightarrow S(F(c, \phi), \emptyset).$$

□

Let F be an idempotent sentential operator. Then we say I is F-closed if either F is accessible (and I is closed under successors) or F is additive and I is closed under addition.

Result

Theorem

Let F be an idempotent sentential operator, $I \subseteq_{\text{end}} M$ be F-closed and separable. Then there is a full satisfaction class S such that $I = \{x : \forall y < x \forall \phi (S(\phi, \emptyset) \leftrightarrow S(F(y, \phi), \emptyset))\}$.

We also say $I \subseteq_{\text{end}} M$ has no least F-gap above it if for each $x > I$, there is $y > I$ such that for each $n \in \omega$, $y \odot n < x$, where $\odot$ is $+$ if F is accessible and $\times$ if F is additive.

Theorem

Let F be an idempotent sentential operator, $I \subseteq_{\text{end}} M$ F-closed and has no least F-gap above it. Then there is a full satisfaction class S such that $I = \{x : \forall y < x \forall \phi (S(\phi, \emptyset) \leftrightarrow S(F(y, \phi), \emptyset))\}$.

Converse

Proposition

Let F be an accessible idempotent sentential operator, S a full satisfaction class and

$$I = \{x : \forall y < x \forall \phi (S(\phi, \emptyset) \leftrightarrow S(F(y, \phi), \emptyset))\}.$$

Then either there is no least $\mathbb{Z}$-gap above I or I is separable.

Proof: Suppose $\{c - n : n \in \omega\}$ is the least $\mathbb{Z}$-gap above I. Then there is ϕ such that $\neg S(F(c, \phi), \emptyset) \leftrightarrow S(\phi, \emptyset)$. In fact, for each $x < c$, one has $S(F(x, \phi), \emptyset) \leftrightarrow S(\phi, \emptyset)$ if and only if $x \in I$. Let $D = \{F(x, \phi) \leftrightarrow \phi : x < c\}$; then by our "local" results, $A = \{F(x, \phi) \leftrightarrow \phi : x \in I\}$ is separable.

Converse, II

Proposition

Let F be an additive idempotent sentential operator, S a full satisfaction class and

$$I = \{x : \forall y < x \forall \phi (S(\phi, \emptyset) \leftrightarrow S(F(y, \phi), \emptyset))\}.$$

Then either there is no least $+$-gap above I or I is separable.

(Proof is more involved.)

Arithmetic Saturation, again

Theorem

Let $\mathcal{M}$ be countable, recursively saturated. Then the following are equivalent:

1. *$\mathcal{M}$ is arithmetically saturated.*
2. *For every idempotent sentential operator F and every F-closed cut I, there is a full satisfation class S such that*

$$I = \{x : \forall y < x \forall \phi (S(\phi, \emptyset) \leftrightarrow S(F(y, \phi), \emptyset))\}.$$

(1) $\implies$ (2): if ω is strong, then every cut which is ω-coded is separable. If it has a least F-gap, it is ω-coded!

(2) $\implies$ (1): if ω is not strong, then cuts which have least F-gaps are not separable. Previous slides: these cuts cannot be these "F-correct" cuts.

Thank you!

The results mentioned today will appear in Abdul-Quader and Łełyk, "Pathologies in satisfaction classes." (Work in progress)

Some other references:

- Cieśliński, Łełyk and Wcisło, The two halves of disjunctive correctness. Journal of Mathematical Logic (in press).
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